$p$-ADIC AND ADELIC GRAVITY AND COSMOLOGY

Branko Dragovich
http://www.phy.bg.ac.yu/~dragovich
dragovich@ipb.ac.rs

Institute of Physics
University of Belgrade
Belgrade, Serbia

ISAAC2013: 9th International Congress
Session 22: Analytic Methods in Complex Geometry
August 5-9, 2013
Cracow
1 Introduction
2 Numbers and their applications
   - $p$-adic numbers, adeles and their functions
   - rational numbers and measurements
   - $p$-adic numbers, adeles and their applications
3 $p$-Adic and adelic gravity
   - $p$-adic and adelic string theory
   - $p$-adic and adelic Einstein gravity
4 $p$-Adic and adelic cosmology
   - $p$-adic and adelic quantum cosmology
   - $p$-adic origin of dark matter and dark energy?
5 Concluding remarks
1. Introduction: $p$-adic mathematical physics (1987)

- $p$-adic and adelic string theory
- $p$-adic and adelic quantum mechanics and (quantum) field theory
- $p$-adic and adelic gravity and (quantum) cosmology
- $p$-adic and adelic space-time structure at the Planck scale
- $p$-adic and adelic dynamical systems
- $p$-adic stochastic processes
- $p$-adic aspects of information theory
- $p$-adic structure of the genetic code
- $p$-adic protein dynamics
- $p$-adic wavelets, ...
1. Introduction: international conferences on $p$-adic mathematical physics

- Workshop on $p$-adic methods in modeling complex systems, Bielefeld, April 2013.
discovered by Kurt Hensel (1861-1941) in 1897.

Any $p$-adic number ($x \in \mathbb{Q}_p$) has a unique canonical representation

$$x = p^{\nu(x)} \sum_{n=0}^{+\infty} x_n p^n, \; \nu(x) \in \mathbb{Z}, \; x_n \in \{0, 1, \ldots, p-1\}$$
2. $p$-Adic numbers, adeles and their functions

- $p$-adic numbers have not natural ordering
- $p$-adic numbers cannot be completely visualized in real Euclidean space: partial visualization by trees and fractals

All triangles are isosceles.
There is no partial intersection of balls.
Any point of a ball is its center.
2. $p$-Adic numbers, adeles and their functions

- Ostrowski theorem: Any nontrivial norm on $\mathbb{Q}$ is equivalent to usual absolute value or to $p$-adic norm, where $p$ is any prime number ($p = 2, 3, 5, 7, 11, \ldots$).
- $p$-adic norm of $x \in \mathbb{Q}$: $|x|_p = |p^{\nu \frac{a}{b}}|_p = p^{-\nu}$, $\nu \in \mathbb{Z}$.
- $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ and $\mathbb{R}$.
- Completion of $\mathbb{Q}$ with respect to $p$-adic distance gives the field $\mathbb{Q}_p$ of $p$-adic numbers, in analogous way to construction of the field $\mathbb{R}$ of real numbers.
2. $p$-Adic numbers, adeles and their functions

- There are mainly two kinds of analysis of $p$-adic variable:
  - (i) $p$-adic valued functions of $p$-adic variable
  - (ii) complex (real) valued functions of $p$-adic variable.

- Usual complex-valued functions of $p$-adic argument are:

\[
\begin{align*}
\pi_s(x) &= |x|^s_p, \\
\chi_p(x) &= \exp 2\pi i \{x\}_p, \\
\Omega(|x|_p) &= \begin{cases} 
1, & |x|_p \leq 1 \\
0, & |x|_p > 1.
\end{cases}
\end{align*}
\]

- Analysis of complex (real) valued functions of $p$-adic (and real) variables is necessary to connect models with measurements.
Real and $p$-adic numbers are unified by adeles. An adele $\alpha$ is an infinite sequence

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R}, \quad \alpha_p \in \mathbb{Q}_p$$

where for all but a finite set $\mathcal{P}$ of primes $p$ one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, i.e. $p$-adic integers.

Space of adeles

$$\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p.$$
2. $p$-Adic numbers, adeles and their functions

Connection of $p$-adic and real properties of rational numbers

\[ |x|_\infty \times \prod_{p \in \mathbb{P}} |x|_p = 1, \text{ if } x \in \mathbb{Q}^\times \]

\[ \chi_\infty(x) \times \prod_{p \in \mathbb{P}} \chi_p(x) = 1, \text{ if } x \in \mathbb{Q} \]

\[ \chi_\infty(x) = \exp(-2\pi ix), \quad \chi_p(x) = \exp 2\pi i\{x\}_p \]
2. Rational numbers and measurements

- Any measurement can be viewed as measurement of a distance.
- Measurement of a distance is comparison of its length with unit length.
- Result of measurement is a rational number with an error.
- This rational number is a real, but not a $p$-adic, number.
- Measurement is related to the Archimedes axiom in geometry.
- According to quantum gravity one cannot measure distances smaller than the Planck length:
  \[ \Delta x \geq \ell_P = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33} \text{cm} \]
  This is limit for application of real numbers in micro-world and a window for application of $p$-adic numbers and adeles.
2. \( p \)-Adic numbers, adeles and their applications

Applications must contain complex (real) valued functions of \( p \)-adic (or adelic) argument. At a hidden level applications can also contain \( p \)-adic (or adelic) valued functions of \( p \)-adic (or adelic) valued arguments.

- At the Planck scale and string theory.
- Gravity theory and cosmology.
- In bioinformation systems – genetic code, ....
- In very complex systems with hierarchical structure.
- Some other applications.
3. $p$-Adic and adelic string theory

Volovich, Vladimirov, Freund, Witten, Dragovich, ...

Electron ($e^-$) and positron ($e^+$) scattering in quantum field theory and string theory.
3. \( p \)-Adic and adelic string theory

String amplitudes:

- standard crossing symmetric Veneziano amplitude

\[
A_{\infty}(a, b) = g_{\infty}^2 \int_{\mathbb{R}} |x|_{\infty}^{a-1} |1 - x|_{\infty}^{b-1} d_{\infty} x
\]

\[
= g_{\infty}^2 \frac{\zeta(1 - a)}{\zeta(a)} \frac{\zeta(1 - b)}{\zeta(b)} \frac{\zeta(1 - c)}{\zeta(c)}
\]

- \( p \)-adic crossing symmetric Veneziano amplitude

\[
A_p(a, b) = g_p^2 \int_{\mathbb{Q}_p} |x|_{p}^{a-1} |1 - x|_{p}^{b-1} d_p x
\]

\[
= g_p^2 \frac{1 - p^{a-1}}{1 - p^{-a}} \frac{1 - p^{b-1}}{1 - p^{-b}} \frac{1 - p^{c-1}}{1 - p^{-c}}
\]

where \( a = -s/2 - 1 \) and \( a, b, c \in \mathbb{C} \) and \( a + b + c = 1 \).
Freund-Witten product formula for adelic strings

\[ A(a, b) = A_\infty(a, b) \prod_p A_p(a, b) = g_\infty^2 \prod_p g_p^2 = \text{const.} \]

Amplitude for real string \( A_\infty(a, b) \), which is a special function, can be presented as product of inverse \( p \)-adic amplitudes, which are elementary functions.
There is an effective field description of scalar open and closed $p$-adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher (Koba-Nielsen) ones at the tree-level.

The exact tree-level Lagrangian for effective scalar field $\varphi$ which describes open $p$-adic string tachyon is

$$\mathcal{L}_p = \frac{m_p D}{g_p^2} \frac{p^2}{p - 1} \left[ -\frac{1}{2} \varphi p^{-\frac{\Box}{2m_p^2}} \varphi + \frac{1}{p + 1} \varphi^{p+1} \right]$$

where $p$ is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the $D$-dimensional d’Alembertian and metric with signature $(- + \ldots +)$ (Freund, Witten, Frampton, Okada, ...).
Einstein Theory of Gravity (ETG) and Quantum Theory lie in foundation of modern theoretical physics.

General theory of relativity is Einstein theory of gravity.

At the cosmic scale there is only gravitational force (interaction). Hence gravitational force governs dynamics of the Universe as a whole.

Cosmology is a science about the Universe as a whole.

There are cosmic observational results which have not generally accepted explanation. Two of these observations are: accelerated expansion of the Universe and large velocities of stars in spiral galaxies.
3. \( p \)-Adic and adelic Einstein gravity

- In ETG gravity is presented in terms of the pseudo-Riemannian geometry.
- In (pseudo-)Riemannian geometry distance is defined by \( ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \) with respect to some system of coordinates.
- From metric tensor \((g_{\mu\nu})\) one can obtain any information about Riemannian space.

\[
g_{\mu\nu} \rightarrow \Gamma^\alpha_{\mu\nu} \rightarrow R^\alpha_{\beta\mu\nu} \rightarrow R_{\mu\nu} \rightarrow R
\]

- Christoffel symbol

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})
\]
3. \( \rho \)-Adic and adelic Einstein gravity

- Riemann-Christoffel curvature tensor

\[
R^\alpha_{\beta \mu \nu} = \partial_\mu \Gamma^\alpha_{\beta \nu} - \partial_\nu \Gamma^\alpha_{\mu \beta} + \Gamma^\alpha_{\mu \gamma} \Gamma^\gamma_{\beta \nu} - \Gamma^\alpha_{\nu \gamma} \Gamma^\gamma_{\mu \beta}
\]

- Ricci tensor and Ricci scalar

\[
R_{\mu \nu} = R^\alpha_{\mu \nu \alpha}, \quad R = g^{\mu \nu} R_{\mu \nu}
\]

- Einstein equations for gravity field (metric tensor)

\[
R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \frac{8 \pi G}{c^4} T_{\mu \nu} - \Lambda g_{\mu \nu}
\]

- Einstein-Hilbert action (\( g \) is determinant of \( g_{\mu \nu} \))

\[
S = \frac{1}{16 \pi G} \int \sqrt{-g}(R - 2\Lambda) d^4 x, \quad (c = 1)
\]
One can formally consider main ingredients of (pseudo)Riemannian geometry and Einstein theory of gravity as $p$-adic valued. It has not direct physical meaning, but can be used in argument of the path integrals.

We now consider Quantum Cosmology (QC). In QC we are interested in wave function of the Universe, which contains quantum information about the Universe at its very early stage of evolution.

We use adelic quantum mechanics to obtain adelic wave function of the Universe.

As a simple example we consider the de Sitter model of the Universe.
Adelic space-time and adelic (pseudo)Riemannian geometry

\[ \alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R}, \quad \alpha_p \in \mathbb{Q}_p \]

where for all but a finite set \( P \) of primes \( p \) one has that \( \alpha_p \in \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \} \), i.e. \( p \)-adic integers.

Adele \( \alpha \) can be space-time point or any ingredient of (pseudo)Riemannian geometry, including gravity action.

Space of adeles

\[ \mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p. \]
4. $p$-Adic and adelic quantum cosmology

- Suppose that at the Planck scale ($10^{-33} \text{cm}$) there are real and $p$-adic strings, and that the space-time is adelic. Then it means that the Universe at the very beginning was adelic and it gives rise to consider adelic quantum cosmology.

- Adelic quantum cosmology is application of adelic quantum mechanics to the cosmological models.

- Adelic quantum mechanics can be viewed as a triple $(L_2(\mathbb{A}), W, U(t))$, where $L_2(\mathbb{A})$ is the Hilbert space on $\mathbb{A}$, $W$ means Weyl quantization, and $U(t)$ is unitary evolution operator on $L_2(\mathbb{A})$.

- $U(t)$ can be expressed through its kernel $K(x'', t''; x', t')$

$$\Psi_p(x'', t'') = \int K(x'', t''; x', t')\Psi_p(x', t')dx',$$
where

$$\Psi_P(x'', t'') = \psi_\infty(x_\infty, t_\infty) \prod_{p \in \mathcal{P}} \psi_p(x_p, t_p) \prod_{p \notin \mathcal{P}} \Omega(|x_p|_p)$$

is adelic eigenfunction, and

$$\mathcal{K}(x'', t''; x', t') = \mathcal{K}_\infty(x''_\infty, t''_\infty; x'_\infty, t'_\infty) \prod_p \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p)$$

$$= \prod_v \mathcal{K}_v(x''_v, t''_v; x'_v, t'_v)$$

is adelic transition probability amplitude.
$\mathcal{K}(x''', t'''; x', t')$ is naturally realized by Feynman’s path integral

\[
\mathcal{K}(x''', t'''; x', t') = \int \chi_\mathbb{A} \left( -\frac{1}{\hbar} S_\mathbb{A}[q] \right) \mathcal{D}_\mathbb{A} q
\]

\[
= \prod_v \int \chi_v \left( -\frac{1}{\hbar} \int_{t'_v}^{t''} L(\dot{q}_v, q_v) \, dt_v \right) \mathcal{D} q_v
\]

Vacuum state $\Omega(|x''_p\rangle_p)$ has property

\[
\Omega(|x''_p\rangle_p) = \int_{\mathbb{Q}_p} \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p) \Omega(|x'_p\rangle_p) \, dx'_p = \int_{\mathbb{Z}_p} \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p) \, dx'_p
\]
Some interesting results of adelic quantum modeling:

- Connection between adelic harmonic oscillator and Riemann zeta function
- Discreteness of space and time at the Planck scale.
The Einstein-Hilbert action for the de Sitter model of the Universe is

\[ S = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{hK} \]

Large scale space is **homogenous** and **isotropic**, and described by the FLRW metric

\[
\begin{align*}
    ds^2 &= \sigma^2 \left[ -N^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \right] \\
    ds^2 &= \sigma^2 \left[ -\frac{N^2}{q(t)} dt^2 + q(t) d\Omega_3^2 \right], \quad \sigma^2 = \frac{2G}{3\pi}
\end{align*}
\]
The corresponding action is

\[ S_v[q] = \frac{1}{2} \int_{t'}^{t''} dtN \left( -\frac{\dot{q}^2}{4N^2} - \lambda q + 1 \right) , \quad \lambda = \frac{2\Lambda G}{9\pi} \]

For \( N = 1 \), the equation of motion \( \ddot{q} = 2\lambda \) has solution

\[ q(t) = \lambda t^2 + \left( \frac{q'' - q'}{T} - \lambda T \right) t + q' , \]

which presents classical trajectory, where \( q'' = q(t'') \), \( q' = q(t') \), \( T = t'' - t' \). This solution resembles the motion of a particle in a constant field.
4. *p*-Adic and adelic quantum cosmology

- The corresponding action for classical trajectory (path) is

\[
\bar{S}_v(q'', T; q', 0) = \frac{\lambda^2 T^3}{24} - \left[\lambda(q' + q'') - 2\right] \frac{T}{4} - \frac{(q'' - q')^2}{8T}, \quad v = \infty, 2, 3, ...
\]

- The corresponding transition amplitude is

\[
\mathcal{K}_v(q'', T; q', 0) = \frac{\lambda_v(-8T)}{|4T|^{\frac{1}{2}} v} \chi_v(-\bar{S}_v(q'', T; q', 0))
\]

- The corresponding vacuum state \(\Omega(|q_p|)\) follows from

\[
\int_{|q'|_p \leq 1} \mathcal{K}_p(q'', T; q', 0) \, dq' = \Omega(|q''|_p)
\]
The $p$-adic Hartle-Hawking wave function is given by

$$
\psi_p(|q|_p) = \int_{|T|_p \leq 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p \left( -\frac{\lambda^2 T^3}{24} [\lambda q - 2] \frac{T}{4} + \frac{q^2}{8T} \right)
$$

and gives vacuum state $\Omega(|q|_p)$ with the condition

$$
\lambda = 3 \cdot 4 \cdot \ell, \quad \ell \in \mathbb{Z}.
$$

Existence of the vacuum state for all (or almost all primes $p$ is a necessary condition for a model to be adelic).

Any adelic quantum model provides some discreteness. In adelic quantum cosmology length of discreteness is the Planck length.
Discreteness at the Planck scale

\[ \psi(q) = \psi_\infty(q) \prod_p \Omega(|q|_p) = \begin{cases} 
\psi_\infty(q), & q \in \mathbb{Z} \\
0, & q \in \mathbb{Q} \setminus \mathbb{Z}.
\end{cases} \]

It means \( q = n \cdot \ell_P \), where \( \ell_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35} \text{m} \).
If Einstein general theory of relativity is theory of gravity for the entire Universe then there is only about 5% of ordinary matter and 95% of matter in the Universe of unknown nature, i.e. 95% of the Universe is dark.

*Dark matter* (27%) should be responsible for rotation velocities of galaxies.

*Dark energy* (68%) was introduced in 1998 as a possibility to explain Universe expansion acceleration.

It is possible that there is some *p-adic matter* in the Universe, and that dark matter and dark energy have *p-adic origin*, i.e. origin in *p-adic strings.*
In the last decade there are a lot of papers related to modification of the Einstein theory of gravity to find alternative (without dark energy) explanation of accelerated expansion of the Universe.

**Nonlocal modified gravity** is one of attractive approaches to generalization of Einstein theory of gravity, and its motivation is also in $p$-adic string theory.
5. Concluding remarks: Main references

BOOKS
5. Concluding remarks: Main references

**REVIEW ARTICLES**


**SOME PAPERS**


5. Concluding remarks

- Numerical results of experiments are real rational numbers – they are not $p$-adic numbers.

- There are some interesting and promising applications of $p$-adic analysis in $p$-adic mathematical physics (string theory, gravity and cosmology, ...) and biology (genetic code,...).

- We conjecture that $p$-adic numbers will play significant role in some complex systems, like complex numbers play role in quantum theory.